C3M4

Parametric Surfaces

To parameterize a surface we must do two things. We must determine a portion of the plane to serve as the domain of a continuous function. Then we must determine that function that maps this domain onto the surface we are parameterizing in a nice way. Later we will learn why this is of value. You will find that we will lean heavily on $z = f(x, y)$, $y = g(x, z)$, or $x = h(y, z)$ in the rectangular case. But cylindrical or spherical coordinates will be of equal value in accomplishing this task.

If D is a set in $\mathbb{R}^2$ and g is defined as

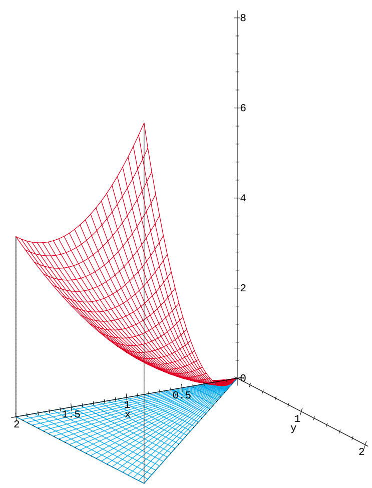
$$g\colon D \to \mathbb{R}^3 \qquad g(u, v) = \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \\ g_3(u, v) \end{pmatrix}$$

then our surface is defined as the image $g(D) = S$. This looks worse than it really is. For one thing, the x, y, z outside of the parentheses do not normally go there. They were put there this time to emphasize that the entry for that row determines the value of the x, y, z coordinate.

Suppose $z = f(x, y)$ with domain D. Define $g(u, v) = \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}$ where it is obvious that u and v play the role of x and y. Or, just use x and y as the independent variables. After a few examples this will seem easier. Note how the parameterization ties in nicely with the plotting of the surface. While we could easily use *cylinderplot* or *sphereplot* in certain problems, having the function g makes it much easier to just use *plot3d*. One of the tricks to ease parameterization is to ask yourself this question: "Is the surface constant for any of the variables in any of the three coordinate systems that we use?" If so, using the other two variables is probably the easiest way to proceed.

In each example that follows we will define the function in Maple and then use *plot3d* to display it. The first step makes us get used to the idea that we are defining a function and the second forces us to define the domain of the function. When we set up the *plot3d* restrictions on the variables we are defining the domain. These skills will be essential later in the course.

Example 1 Parameterize the portion of the surface $z = x^2 + y^2$ that lies above the triangle with vertices $P(0, 0)$, $Q(2, 0)$, $R(2, 2)$. Because $z = f(x, y)$, we just use the obvious approach as described above.



$$g(x, y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix} \qquad \begin{array}{l} 0 \le y \le x \\ 0 \le x \le 2 \end{array}$$

Define the function in Maple:

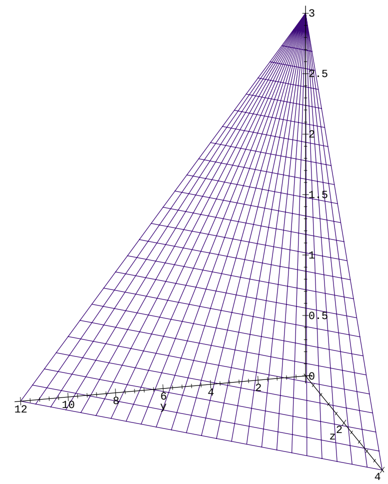
```
> g:=(x,y)->[x,y,x^2+y^2];
```

$$g := (x, y) \to [x, y, x^2 + y^2]$$

To plot this surface use:

```
> plot3d(g(x,y),x=0..2,y=0..x,color=red);
```

Example 2 Parameterize that portion of the plane $x + 3y + 4z = 12$ that lies in the first octant with x as the dependent variable.



$$g(y,z) = \begin{pmatrix} 12 - 3y - 4z \\ y \\ z \end{pmatrix} \qquad \begin{matrix} 0 \le y \le 4 - \frac{4z}{3} \\ 0 \le z \le 3 \end{matrix}$$

Define the function in Maple:

```
> g:=(y,z)->[12-3*y-4*z,y,z];
```

$$g := (y, z) \to [12 - 3y - 4z, y, z]$$

And to plot this surface:

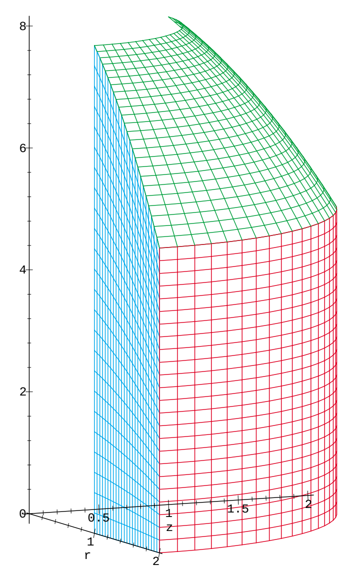
```
> plot3d(g(y,z),y=0..4-4*z/3,z=0..3,color=blue);
```

You may be puzzled by this plot. Pay attention to the axes. Because the bounds on y were listed first, Maple thinks that y belongs where we put the x-axis. It is very important that you realize that the domain of this function g is the triangle in the yz-plane bounded by the y and z axes and the plane $x+3y+4z = 12$.

Before we do this next example we remind you of the cylindrical coordinate system.

$$\begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

Example 3 The solid in the first octant lies between $x^2+y^2 = 1$ and $x^2 + y^2 = 4$, above $z = 0$ and below the paraboloid $z = 9-x^2-y^2$. We are going to parameterize the surfaces that we can see and show how to plot them in Maple. This is easiest when done from the viewpoint of cylindrical coordinates. We will begin with the surface on the left, where $y = 0$ or $\theta = 0$. Note that having $\theta = 0$ lets us use r and z. Observe that $\cos 0 = 1$ and $\sin 0 = 0$.



$$g1(r,z) = \begin{pmatrix} r \\ 0 \\ z \end{pmatrix} \qquad \begin{matrix} 1 \le r \le 2 \\ 0 \le z \le 9 - r^2 \end{matrix}$$

The function that parameterizes the top is:

$$g2(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 9 - r^2 \end{pmatrix} \qquad \begin{matrix} 1 \le r \le 2 \\ 0 \le \theta \le \pi/2 \end{matrix}$$

The domain is the annular region between the circles in the first quadrant.

Now we will parameterize the outside wall. In this case, r is constant. What determines the upper bound on z?

$$g3(\theta,z) = \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ z \end{pmatrix} \qquad \begin{matrix} 0 \le \theta \le \pi/2 \\ 0 \le z \le 5 \end{matrix}$$

Define the functions in Maple:

```
> g1:=(r,z)->[r,0,z]
```

$$g1 := (r, z) \to [r, 0, z]$$

```
> g2:=(r,t)->[r*cos(t),r*sin(t),9-r^2]
```

$$g2 := (r, t) \to [r\cos(t), r\sin(t), 9 - r^2]$$

```
> g3:=(t,z)->[2*cos(t),2*sin(t),z]
```

$$g3 := (t, z) \to [2\cos(t), 2\sin(t), z]$$

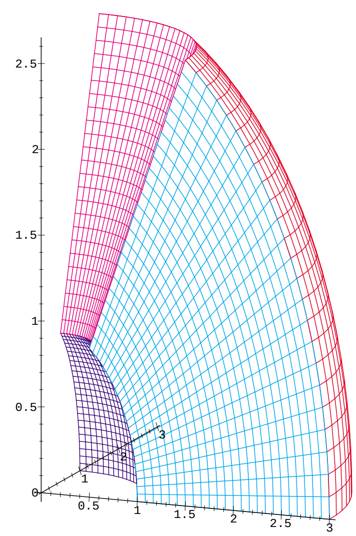
And to plot the surfaces:

```
> A1:=plot3d(g1(r,z),r=1..2, z=0..9-r^2,color=cyan):
> A2:=plot3d(g2(r,t),r=1..2, t=0..Pi/2,color=magenta):
> A3:=plot3d(g3(t,z), t=0..Pi/2, z=0..5,color=red):
> display(A1,A2,A3);
```

Before we do this next example we remind you of the spherical coordinate system.

$$\begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{pmatrix}$$

Example 4 The solid in the first octant lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$, $(x \geq 0, y \geq 0, z \geq 0)$ and below the cone $z = \sqrt{3(x^2 + y^2)}$. We are going to parameterize the four surfaces that we can see in the figure, beginning with the small spherical surface to the lower left where $\rho = 1$. The cone on the top is produced by letting $\varphi = \pi/6$.



$$h1(\theta, \varphi) = \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix} \qquad \begin{array}{c} 0 \leq \theta \leq \pi/2 \\ \pi/6 \leq \varphi \leq \pi/2 \end{array}$$

By adjusting ρ to be 3 we have the outside spherical surface.

$$h2(\theta, \varphi) = \begin{pmatrix} 3\sin\varphi\cos\theta \\ 3\sin\varphi\sin\theta \\ 3\cos\varphi \end{pmatrix} \qquad \begin{array}{c} 0 \leq \theta \leq \pi/2 \\ \pi/6 \leq \varphi \leq \pi/2 \end{array}$$

The side closest to the viewer occurs when $\theta = 0$, $\cos 0 = 1$, $\sin 0 = 0$.

$$h3(\rho, \varphi) = \begin{pmatrix} \rho\sin\varphi \\ 0 \\ \rho\cos\varphi \end{pmatrix} \qquad \begin{array}{c} 1 \leq \rho \leq 3 \\ \pi/6 \leq \varphi \leq \pi/2 \end{array}$$

The top comes from the cone, $\varphi = \pi/6$, where $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$.

$$h4(\rho, \theta) = \begin{pmatrix} \rho(1/2)\cos\theta \\ \rho(1/2)\sin\theta \\ \rho(\sqrt{3}/2) \end{pmatrix} \qquad \begin{array}{c} 1 \leq \rho \leq 3\pi \\ 0 \leq \theta \leq \pi/2 \end{array}$$

We show $h4$ in an unsimplified form to make the process more transparent. Now we translate this into Maple and show how to plot the surfaces.

```
> h1:=(t,phi)->[sin(phi)*cos(t),sin(phi)*sin(t),cos(phi)];
```

$$h1 := (t, \phi) \to [\sin(\phi)\cos(t), \sin(\phi)\sin(t), \cos(\phi)]$$

```
> h2:=(t,phi)->[3*sin(phi)*cos(t),3*sin(phi)*sin(t),3*cos(phi)];
```

$$h2 := (t, \phi) \to [3\sin(\phi)\cos(t), 3\sin(\phi)\sin(t), 3\cos(\phi)]$$

```
> h3:=(rho,phi)->[rho*sin(phi),0,rho*cos(phi)];
```

$$h3 := (\rho, \phi) \to [\rho \sin(\phi), 0, \rho \cos(\phi)]$$

```
> h4:=(rho,t)->[rho*(1/2)*cos(t),rho*(1/2)*sin(t),rho*(sqrt(3)/2)];
```

$$h4 := (\rho, t) \to [\rho \left(\frac{1}{2}\right) \cos(t), \rho \left(\frac{1}{2}\right) \sin(t), \rho \left(\frac{\sqrt{3}}{2}\right)]$$

Now for the plot which appears above:

```
> B1:=plot3d(h1(t,phi),t=0..Pi/2,phi=Pi/6..Pi/2,color=blue):
> B2:=plot3d(h2(t,phi),t=0..Pi/2,phi=Pi/6..Pi/2,color=red):
> B3:=plot3d(h3(rho,phi),rho=1..3,phi=Pi/6..Pi/2,color=cyan):
> B4:=plot3d(h4(rho,t),rho=1..3,t=0..Pi/2,color=magenta):
> display(B1,B2,B3,B4);
```

C3M4 Problems Use Maple to parameterize and to plot using *plot3d*:

1. S, the cylinder $x^2 + y^2 = 3$, for $0 \leq z \leq 2$ and its top
2. T, the triangular plate $3x + y + 4z = 12$ in the first octant $(0 \leq x, 0 \leq y, 0 \leq z)$
3. U, the portion of the sphere of radius 3 that lies above the cone $x^2 + y^2 = z^2$, and in the half-plane $y \geq 0$. And include the cone with $y \geq 0$ and $z \geq 0$.